DEFINITION C.8 *Let $f : \Re^n \to \Re$ be a convex function defined on $\Re^n$. A* **subgradient** *of f at point $\mathbf{x}$ is a vector $\mathbf{d} \in \Re^n$ satisfying*

$$f(\mathbf{y}) \geq f(\mathbf{x}) + \mathbf{d}^\top(\mathbf{y} - \mathbf{x}), \quad \forall \mathbf{y} \in \Re^n.$$

The *subdifferential* of a f at $\mathbf{x}$, denoted $\partial f(\mathbf{x})$, is defined as the set of all subgradients of f at $\mathbf{x}$. The definition for concave functions simply has the above inequality reversed.

Optimization Problems

Let $\mathbf{x} \in \Re^n$ denote a vector of *decision variables*, $f : \Re^n \to R$ be a given *objective function* and $X \subseteq \Re^n$ be a *constraint set.* A point $\mathbf{x}$ in the set X is called *feasible,* and points not in X are called *infeasible.* In a *maximization problem,* we seek a feasible solution $\mathbf{x}^*$—called a *global maximum* (or *global optimum*)—such that

$$f(\mathbf{x}^*) \geq f(\mathbf{x}), \ \forall \mathbf{x} \in X. \tag{C.2}$$

Equivalently, $\mathbf{x}^*$ solves

$$\max_{\mathbf{x} \in X} f(\mathbf{x}). \tag{C.3}$$

We say such an $\mathbf{x}^*$ is *globally optimal.* If X is the empty set, then the above optimization problem is said to be *infeasible;* otherwise, the problem *is feasible.* If $X = \Re^n$, the problem is said to be *unconstrained.* The problem is called *unbounded* if there exists a sequence of feasible points $\{\mathbf{x}^{(k)}; k = 1, 2, \ldots\}$ with $\mathbf{x}^{(k)} \in X$ for all k and $\lim_{k \to \infty} f(\mathbf{x}^{(k)}) = +\infty$.

A minimization problem reverses the inequality above and is equivalent to maximizin $-f(\mathbf{x})$. We focus here on only the maximization version. If f is concave and X is convex, then the problem (C.3) is called a *convex optimization problem.*

Let $N(\mathbf{x}, \epsilon) = \{\mathbf{y} : \|\mathbf{y} - \mathbf{x}\| \leq \epsilon\}$ denote the ball of radius ϵ about the point $\mathbf{x}$. A solution $\mathbf{x}^*$ is called a *local maximum* (or *local optimum*) if there exists an $\epsilon > 0$ such that

$$f(\mathbf{x}^*) \geq f(\mathbf{x}), \ \forall \mathbf{x} \in X \cap N(\mathbf{x}^*, \epsilon).$$

We say such an $\mathbf{x}^*$ is *locally optimal.* Note all global optima are also locally optimal.

In the convex case, local and global optima coincide:

PROPOSITION C.5 *If f is a concave function defined on a convex set X, then any local maximum is a global maximum. If f is strictly concave, then if a global maximum exists, it is unique.*

Optimality Conditions

Optimality conditions help identify and characterize optimal solutions. They are useful both theoretically and computationally.

Suppose $f \in C^1$. Then we have the following *first-order necessary conditions* for $\mathbf{x}^*$ to be an optimal solution:

PROPOSITION C.6 *If $f \in C^1$ and $\mathbf{x}^*$ is a local maximum, then there exists an $\epsilon > 0$ such that*

$$\nabla f(\mathbf{x}^*)^\top(\mathbf{x} - \mathbf{x}^*) \leq 0, \ \forall \mathbf{x} \in X \cap N(\mathbf{x}^*, \epsilon).$$

In particular, if $X = \Re^n$ (the unconstrained case), then this condition reduces to

$$\nabla \mathbf{f}(\mathbf{x}^*) = 0. \tag{C.4}$$